To: PFJL
From: PJH
Date: November 1, 2006 – 10:53 pm
Re: Quick summary

Basic Formulation

The basic formulation comes from Dukowicz & Smith (1994). The primitive equation momentum equations and conservation of mass, in Cartesian form, are

$$\frac{Du}{dt} - fv = -\frac{1}{\rho_0}\partial_x p + F^x \quad , \tag{1}$$

$$\frac{Dv}{dt} + fu = -\frac{1}{\rho_0}\partial_y p + F^y \quad , \tag{2}$$

$$\partial_z p = -\rho g \ , \tag{3}$$

$$\partial_x u + \partial_y v + \partial_z w = 0 \ , \tag{4}$$

where (u, v, w) are the three components of velocity, p is the total pressure, ρ_0 is a mean density of seawater, ρ is the variable density of seawater, g is the acceleration due to gravity and F contains the sub-gridscale turbulent terms. D&S then decompose the total pressure into a surface pressure, p_s, evaluated at $z = 0$ and a hydrostatic pressure, p_h, evaluated from equation (3)

$$p(x, y, z) = p_s(x, y) + p_h(x, y, z) \quad , \tag{5}$$

$$p_h(x, y, z) = \int_z^0 \rho(x, y, \zeta) g \, d\zeta \quad . \tag{6}$$

The internal components are evaluated as in the Bryan-Cox model. To solve the external components, including the surface pressure, average equations (1) & (2) and integrate (4) all in the vertical:

$$\partial_t U - fV = -\partial_x \pi_s + G^x \quad , \tag{7}$$

$$\partial_t V + fU = -\partial_y \pi_s + G^y \quad , \tag{8}$$

$$\partial_x HU + \partial_y HV + \partial_t \eta = 0 \quad , \tag{9}$$

where (U, V) are the components of the vertically averaged velocity, π_s is the kinematic surface pressure, η is the surface elevation and G now contains the advection and hydrostatic pressure terms in addition to the sub-gridscale representation. Using hydrostatics, the surface pressures can be related to the surface elevation as

$$\pi_s \equiv \frac{1}{\rho_0} p_s = g\eta \quad . \tag{10}$$

Next, D&S introduce a particular time discretization, which is simplified here following their stability conclusions

$$\frac{U^{n+1} - U^{n-1}}{2\Delta t} - fV^{\alpha} = -\partial_x \pi_s^{\alpha} + G^{x,n} + O\left((\Delta t)^2\right) \quad , \tag{11}$$

$$\frac{V^{n+1} - V^{n-1}}{2\Delta t} + fU^{\alpha} = -\partial_y \pi_s^{\alpha} + G^{y,n} + O\left((\Delta t)^2\right) \quad , \tag{12}$$

$$\frac{\pi_s^{n+1} - \pi_s^{n}}{g\Delta t} + \partial_x HU^{n+1} + \partial_y HV^{n+1} = 0 \quad , \tag{13}$$

where equation (10) has been used to replace η by π_s and the superscript α refers to the semi-implicit time discretization

$$U^{\alpha} = \alpha U^{n+1} + (1 - 2\alpha)U^{n} + \alpha U^{n-1} \quad . \tag{14}$$

The momentum equations (11) & (12) are recast in vector form, and rewritten to group all the advanced-time terms on one side of the resulting equation:

$$\left(\mathbf{I} + 2\alpha\Delta t\mathbf{B}\right)\vec{U}^{n+1} + 2\alpha\Delta t\nabla\pi_s^{n+1} = \vec{\mathcal{F}}^{n,n-1} + O\left((\Delta t)^3\right) \quad , \tag{15}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{B}$ is the coriolis matrix

$$\mathbf{B} = \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \quad , \tag{16}$$

and $\vec{\mathcal{F}}^{n,n-1}$ collects all the explicitly known quantities. To decouple the solution for π_s^{n+1} from the solution for $\vec{U}^{n+1}$, D&S split the operator in (15) by introducing the augmented velocity, $\vec{\hat{U}}$,

$$\vec{\hat{U}} = \vec{U}^{n+1} + 2\alpha\Delta t\nabla\delta\pi_s \quad , \tag{17}$$

where we've introduced the notation

$$\delta\pi_s = \pi_s^{n+1} - \pi_s^{n-1} \quad . \tag{18}$$

Substituting (17) & (18) into (15) and regrouping yields

$$\begin{aligned} \left(\mathbf{I} + 2\alpha\Delta t\mathbf{B}\right)\vec{\hat{U}} + 2\alpha\Delta t\nabla\pi_s^{n-1} &= \vec{\mathcal{F}}^{n,n-1} + 4\alpha^2(\Delta t)^2\mathbf{B}\nabla\delta\pi_s + O\left((\Delta t)^3\right) \\ &= \vec{\mathcal{F}}^{n,n-1} + O\left((\Delta t)^3\right) \end{aligned} \tag{19}$$

where the term on the right still containing π_s^{n+1} has been neglected for being of the same order as the discretization error, assuming that $\delta\pi_s$ is $O(\Delta t)$ (a necessary assumption for bounded first derivatives).

To generate an equation for $\delta\pi_s$, first average (13) with itself evaluated one time step earlier:

$$\frac{1}{2}\left\{\frac{\pi_s^{n+1}-\pi_s^n}{g\Delta t}+\nabla\cdot H\vec{U}^{n+1}+\frac{\pi_s^n-\pi_s^{n-1}}{g\Delta t}+\nabla\cdot H\vec{U}^n\right\}=$$

$$\frac{\delta\pi_s}{2g\Delta t}+\frac{1}{2}\nabla\cdot\left[H\vec{U}^{n+1}+H\vec{U}^n\right]=0\quad. \tag{20}$$

Substituting for $\vec{U}^{n+1}$ from equation (17) and isolating the $\delta\pi_s$ terms:

$$\nabla\cdot(H\nabla\delta\pi_s)-\frac{\delta\pi_s}{2\alpha g(\Delta t)^2}=\frac{1}{2\alpha\Delta t}\nabla\cdot\left(H\hat{\vec{U}}+H\vec{U}^n\right)\quad. \tag{21}$$

The general solution procedure is then to

(1) solve (19) for $\hat{\vec{U}}$. This can be directly evaluated, based on the output of `clinic.F`, at the velocity points $i_v\in[2,\,(I_{mt}-2)]$, $j_v\in[2,\,(J_{mt}-2)]$. However, for step (2), $\hat{\vec{U}}$ is needed for all active velocity points $i_v\in[1,\,(I_{mt}-1)]$, $j_v\in[1,\,(J_{mt}-1)]$.
(2) solve (21) for $\delta\pi_s$, with suitable Dirichlet boundary values.
(3) solve (18) for π_s^{n+1}. This can be evaluated at all tracer points $i_t\in[1,\,I_{mt}]$, $j_t\in[1,\,J_{mt}]$.
(4) solve (17) for $\vec{U}^{n+1}$. This can be evaluated at all active velocity points.

Some comments on the Elliptical problem

Equation (21), with Dirichlet boundary conditions, is solved using a pre-conditioned conjugate gradient solver in the SPARSKIT package. The standard convergence test for this package is a measure of the reduction in the norm of the residual over all points. Specifically, let r be the residual of the current solver iteration and r_0 be the residual of the initial guess. The iterative solution is considered converged when

$$\|r\|\le\tau_r\|r_0\|+\tau_a$$

where τ_r is the relative tolerence (10^{-12}) and τ_a is the absolute tolerence (10^{-25}). One issue that came up in nesting is that this condition is an integral measure over the entire domain. For nested models the effectively translates into different convergence criteria for different domains. To control this we've introduced a point-wise constraint *in addition to the residual constraint* above. The constraint we chose (with a view to equation 17) is that at every point

$$\left|\frac{\partial\pi_s^k}{\partial x}-\frac{\partial\pi_s^{k-1}}{\partial x}\right|\le\tau_r^g\left|\frac{\partial\pi_s^k}{\partial x}\right|+\tau_a$$

and

$$\left|\frac{\partial\pi_s^k}{\partial y}-\frac{\partial\pi_s^{k-1}}{\partial y}\right|\le\tau_r^g\left|\frac{\partial\pi_s^k}{\partial y}\right|+\tau_a$$

where the superscript k refers to the iteration number and τ_r^g is the relative tolerence for the gradient test (10^{-5} or 10^{-8}).

Provided Boundary Conditions

To extend the range of $\vec{\hat{U}}$, a number of numerical conditions were applied to the vertically averaged momentum forcing[1]. To extend this list, and provide guidance for 2–way nesting, a set of robust provided BCs for the vertically averaged momentum forcing were sought. Introducing the notation

$$\delta \mathcal{G} = \mathcal{G}^{n+1} - \mathcal{G}^{n-1} \tag{23}$$

and using equation (14), the vertically averaged momentum equations (11-12) can be rewritten as

$$\frac{\delta U}{2\Delta t} - \alpha f \delta V + \frac{\partial}{\partial x}\left[\alpha \delta \pi_s + (1-2\alpha)\pi_s^n + 2\alpha \pi_s^{n-1}\right] = \mathcal{R}_x^{n,n-1} \tag{24}$$

$$\frac{\delta V}{2\Delta t} + \alpha f \delta U + \frac{\partial}{\partial y}\left[\alpha \delta \pi_s + (1-2\alpha)\pi_s^n + 2\alpha \pi_s^{n-1}\right] = \mathcal{R}_y^{n,n-1} \quad . \tag{25}$$

Equations (24-25) represent the particular form used in the HOPS PE model. To evaluate the vertically averaged momentum forcing at the boundaries, we use the lefthand sides of (24-25). In the course of experimenting with these BCs a couple of points were found to improve stability:

(1) Only the quantities at time t_{n+1} should be obtained from boundary data[2]. The terms at time t_{n-1} are available in PE balance and should be taken from memory.

(2) For the free surface case, equations (24-25) should be evaluated in "transport space", *i.e.*:

$$\mathcal{R}_x^{n,n-1} = \left[\frac{\delta(\mathcal{H}U)}{2\Delta t} - \alpha f \delta(\mathcal{H}V)\right]\frac{1}{\mathcal{H}^n} + \frac{\partial}{\partial x}\left[\alpha \delta \pi_s + (1-2\alpha)\pi_s^n + 2\alpha \pi_s^{n-1}\right] \tag{26}$$

$$\mathcal{R}_y^{n,n-1} = \left[\frac{\delta(\mathcal{H}V)}{2\Delta t} + \alpha f \delta(\mathcal{H}U)\right]\frac{1}{\mathcal{H}^n} + \frac{\partial}{\partial y}\left[\alpha \delta \pi_s + (1-2\alpha)\pi_s^n + 2\alpha \pi_s^{n-1}\right] \tag{27}$$

where $\mathcal{H} = H + \eta$ includes the free surface elevation.

Nested Boundary Conditions

For nesting in the free surface, the *transport* in the large domain is interpolated to the boundary of the small domain, and substituted into equations (26-27).

Provided-Orlanski Corrections

[1] Actually the first step was recognizing that the boundary conditions belonged here rather than on one of the other intermediate variables.

[2] In the tidal, free surface case, boundary data is taken to mean the superposition of the geostrophic data values and the instantaneous linear tidal values.

Following the algorithm of Perkins *et al.* (1997), corrections to the provided values are obtained by applying the Orlanski algorithm[3]to the difference between the PE model values and the provided values.

For the barotropic velocity (transport), this is only done for the tangential component to the boundary. The correction to the normal component is derived from the correction to the surface pressure, $\Delta\pi_s = g\Delta\eta$, and the barotropic continuity equation

$$\frac{\partial \Delta\eta}{\partial t} + \nabla \cdot \left(\mathcal{H}\Delta\vec{U}\right) = 0 \quad . \tag{28}$$

References

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[3] Unlike Perkins *et al.*, I only use the standard normal component Orlanski equation, not the 2D version.