Riemannian Geometry of Matrix Manifolds for Lagrangian Uncertainty Quantification of Stochastic Fluid Flows

by

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Abstract

This work focuses on developing theory and methodologies for the analysis of material transport in stochastic fluid flows. In a first part, two dominant classes of techniques for extracting Lagrangian Coherent Structures are reviewed and compared and some improvements are suggested for their pragmatic applications on realistic high-dimensional deterministic ocean velocity fields. In the stochastic case, estimating the uncertain Lagrangian motion can require to evaluate an ensemble of realizations of the flow map associated with a random velocity flow field, or equivalently realizations of the solution of a related transport partial differential equation. The Dynamically Orthogonal (DO) approximation is applied as an efficient model order reduction technique to solve this stochastic advection equation. With the goal of developing new rigorous reduced-order advection schemes, the second part of this work investigates the mathematical foundations of the method. Riemannian geometry providing an appropriate setting, a framework free of tensor notations is used to analyze the embedded geometry of three popular matrix manifolds, namely the fixed rank manifold, the Stiefel manifold and the isospectral manifold. Their extrinsic curvatures are characterized and computed through the study of the Weingarten map. As a spectacular by-product, explicit formulas are found for the differential of the truncated Singular Value Decomposition, of the Polar Decomposition, and of the eigenspaces of a time dependent symmetric matrix. Convergent gradient flows that achieve related algebraic operations are provided. A generalization of this framework to the non-Euclidean case is provided, allowing to derive analogous formulas and dynamical systems for tracking the eigenspaces of non-symmetric matrices. In the geometric setting, the DO approximation is a particular case of projected dynamical systems, that applies instantaneously the SVD truncation to optimally constrain the rank of the reduced solution. It is obtained that the error committed by the DO approximation is controlled under the minimal geometric condition that the original solution stays close to the low-rank manifold. The last part of the work focuses on the practical implementation of the DO methodology for the stochastic advection equation. Fully linear, explicit central schemes are selected to ensure stability, accuracy and efficiency of the method. Riemannian matrix optimization is applied for the dynamic evaluation of the dominant SVD of a given matrix and is integrated to the DO time-stepping. Finally the technique is illustrated numerically on the uncertainty quantification of the Lagrangian motion of two bi-dimensional benchmark flows.

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> En qui Tu es Toi, et moi je suis moi, en qui Toi seul es et moi je ne suis pas, en qui il n'y a ni Toi ni moi, à Celui-là je rends hommage!
>
> Abhinavagupta, Twenty Verses on the Great Teaching, *Trad. Lilian Silburn*

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