Developing a nonhydrostatic isopycnal-coordinate ocean model

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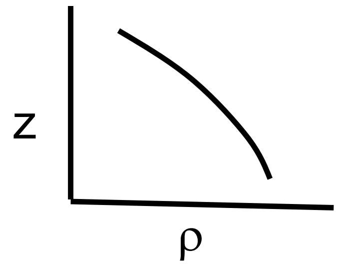
Outline

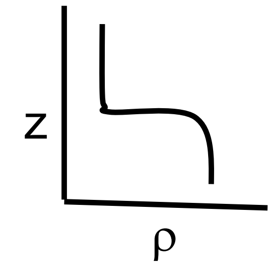
- Overview of internal gravity waves
- Nonhydrostatic (Navier-Stokes) modeling
- Grid-resolution requirements
  →*Nonhydrostatic modeling is expensive!*
- A nonhydrostatic isopycnal-coordinate model
  →*The cost can be reduced!*
- Conclusions

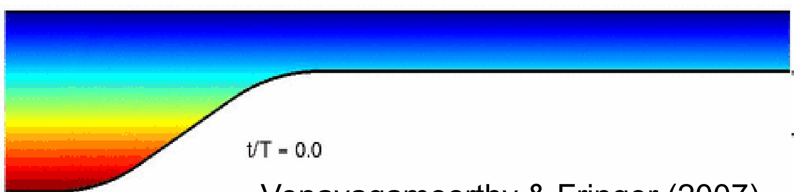
Internal gravity waves

Internal waves

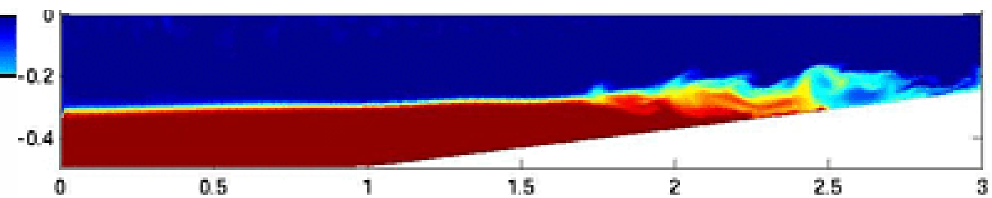


Interfacial waves

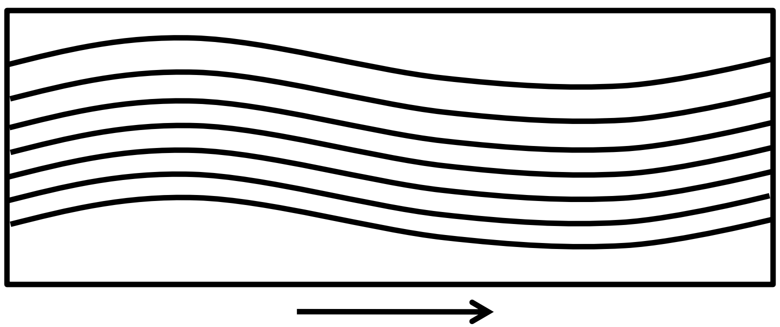


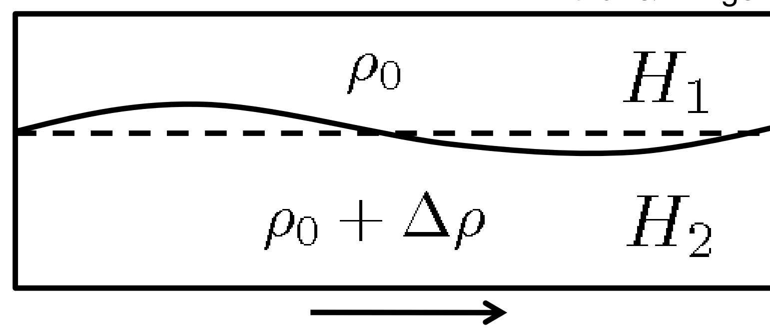


Venayagamoorthy & Fringer (2007)



Arthur & Fringer (2014)





$$c = \frac{ND}{\pi}, N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}$$

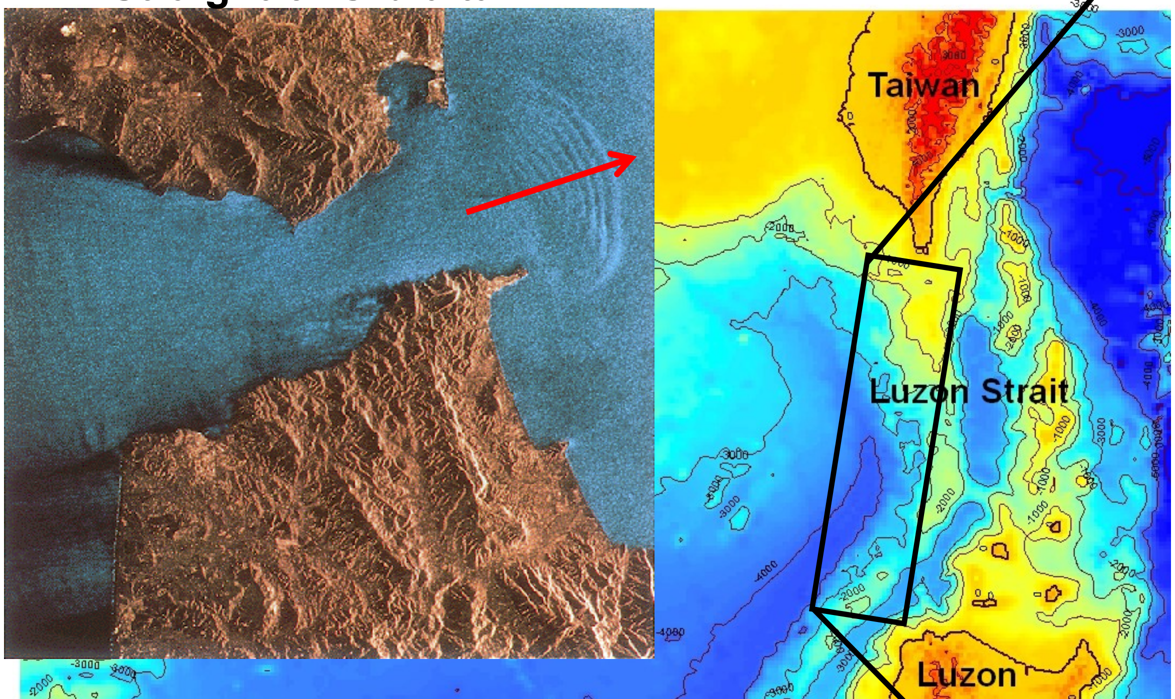
$$c^2 = \frac{g' H_1 H_2}{H_1 + H_2}, g' = \frac{g\Delta\rho}{\rho_0}$$

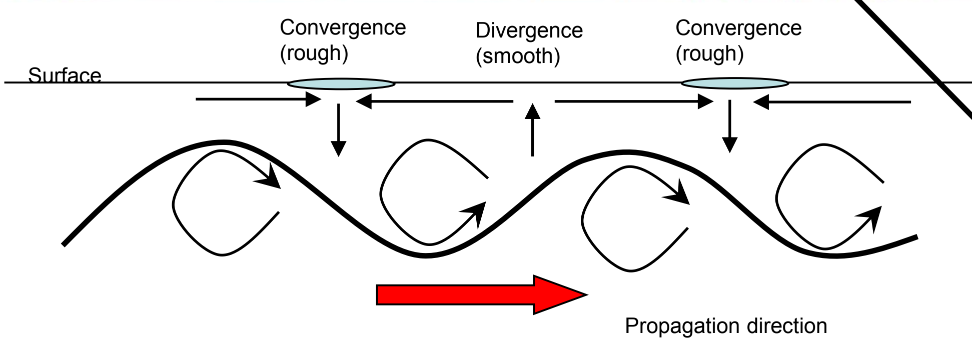
- Typical speeds in the ocean: 1-3 m/s
- Frequencies: Tidal (internal tides) - minutes (internal waves)
- Wavelengths: 100s of km to 10s of m

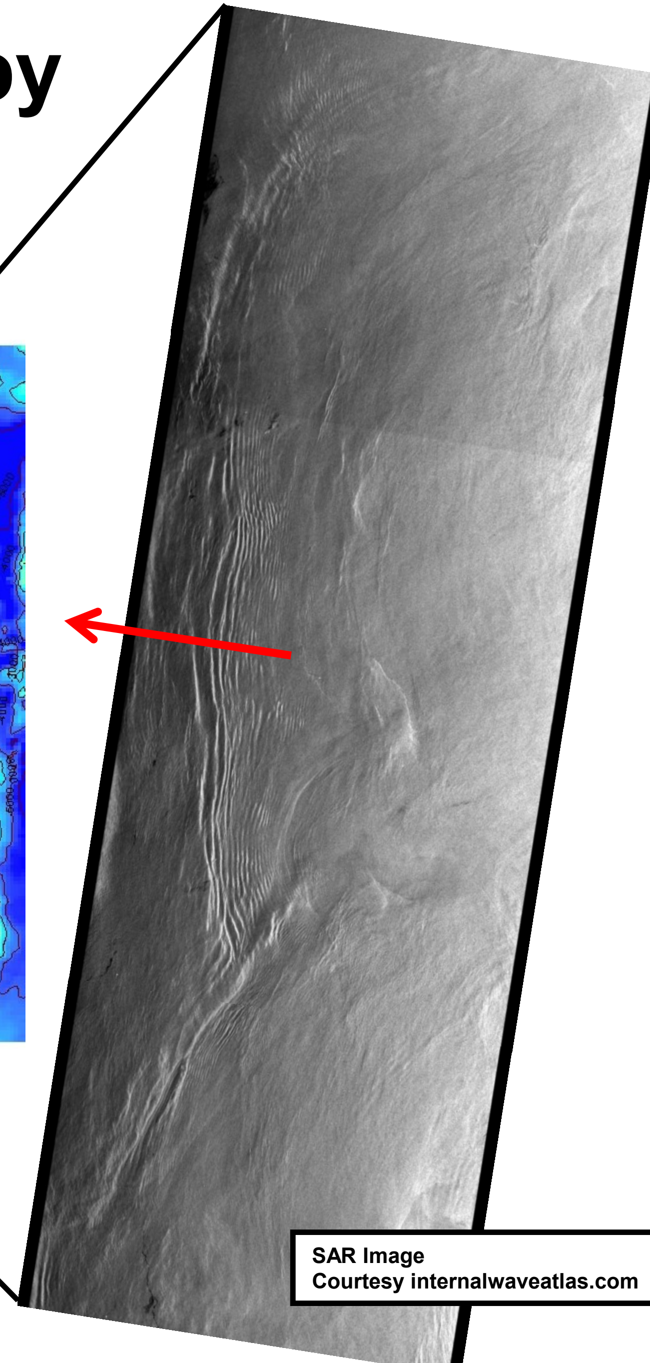
Surface signatures induced by internal gravity waves

Straight of Gibraltar

South China Sea

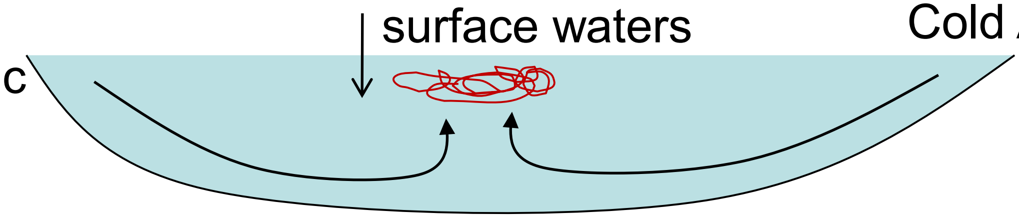






Applications of internal gravity waves

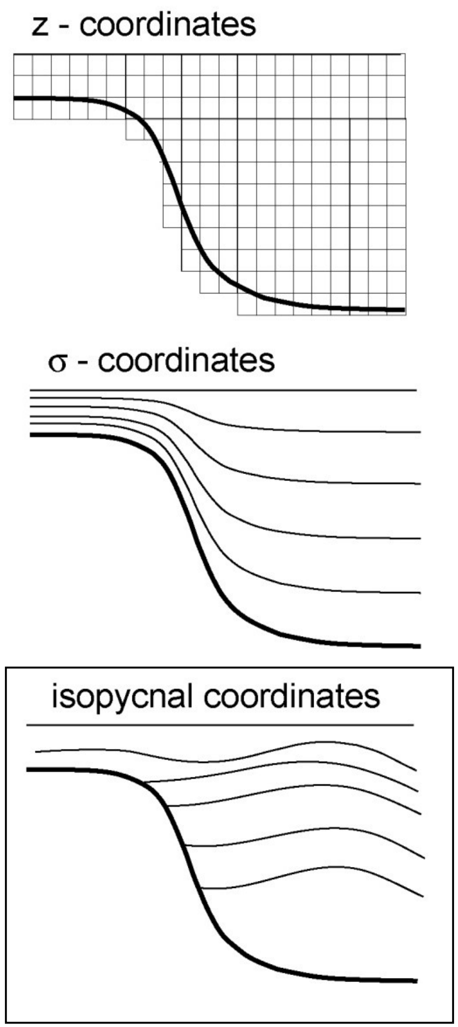
- Breaking of internal tides and waves may provide the necessary mixing to maintain the ocean stratification (Munk and Wunsch 1998).



Mixing induced by breaking internal waves prevents the ocean from turning into a "stagnant pool of cold, salty water"...

- Internal waves are hypothesized to deliver nutrients that sustain thriving coral reef ecosystems (e.g. Florida Shelf, Leichter et al., 2003; Dongsha Atol, Wang et al. 2007)
- Internal waves influence sediment transport in lakes and oceans and propagation of acoustic signals.
- Strong internal wave-induced currents can cause oil platform instability and pipeline rupture.

Isopycnal vs z- or sigma-coordinates



Advantages of isopycnal coordinates:

• Reduces the number of vertical grid points from O(100) in traditional coordinates to O(1-10)

• No spurious vertical (diapycnal) diffusion/mixing

Challenges of isopycnal coordinates:

• Cannot represent unstable stratification

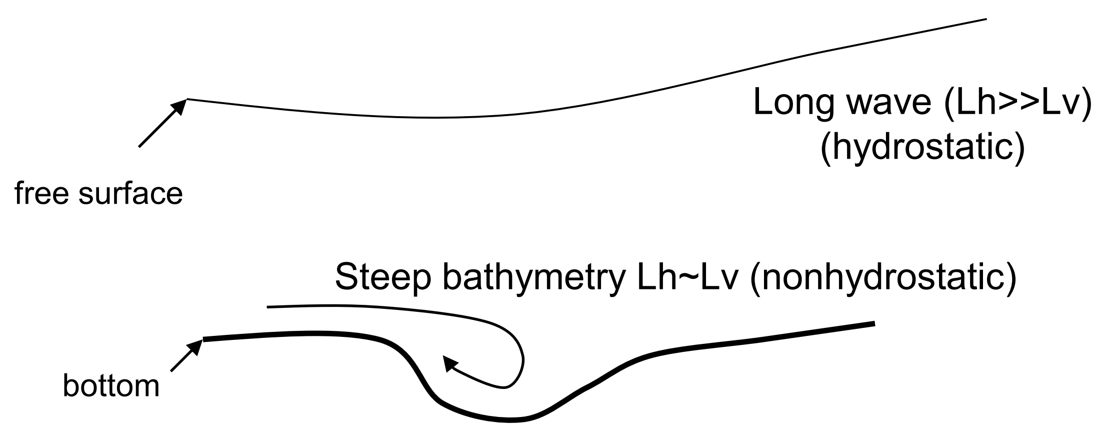
• Layer outcropping (drying of layers) requires special numerical schemes

• Hydrostatic

Hydrostatic vs. nonhydrostatic flows

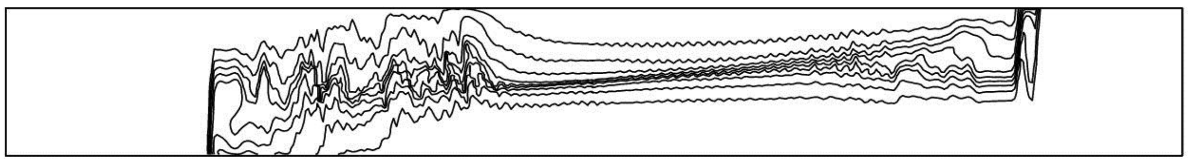
- **Most ocean flows are hydrostatic**
  - Long horizontal length scales relative to vertical length scales, i.e. long waves (i.e. Lh >> Lv)

- **Only in small regions is the flow nonhydrostatic**
  - Short horizontal length scales relative to vertical scales (i.e. Lh ~ Lv)
  - Can cost 10X more to compute!

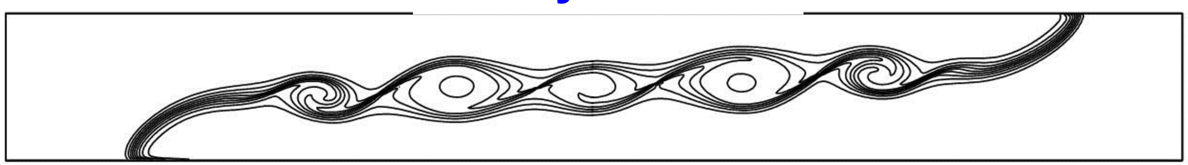


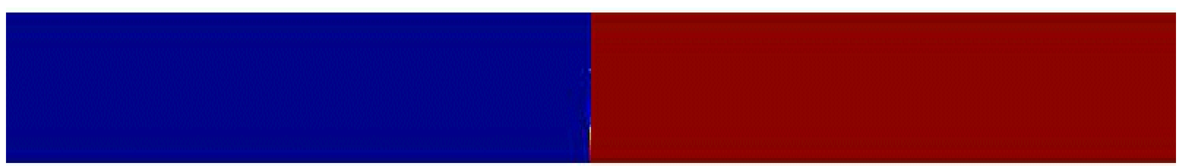
Nonhydrostatic effects: Overturning

Hydrostatic



Nonhydrostatic





Overturning motions and eddies are not the only nonhydrostatic process…

Nonhydrostatic effects: Frequency dispersion of gravity waves

- Dispersion relation for irrotational surface gravity waves:

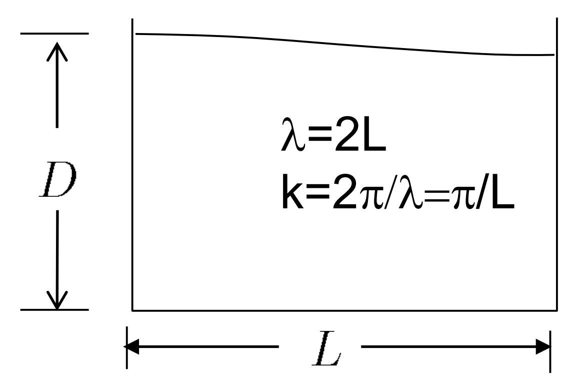
$$c^2 = \frac{g}{k}\tanh kD = \frac{g}{k}\tanh \pi\varepsilon, \quad \varepsilon = \frac{D}{L}$$

- Deep-water limit: $\varepsilon >> 1$ (nonhydrostatic)

$$c^2 = \frac{g}{k}$$

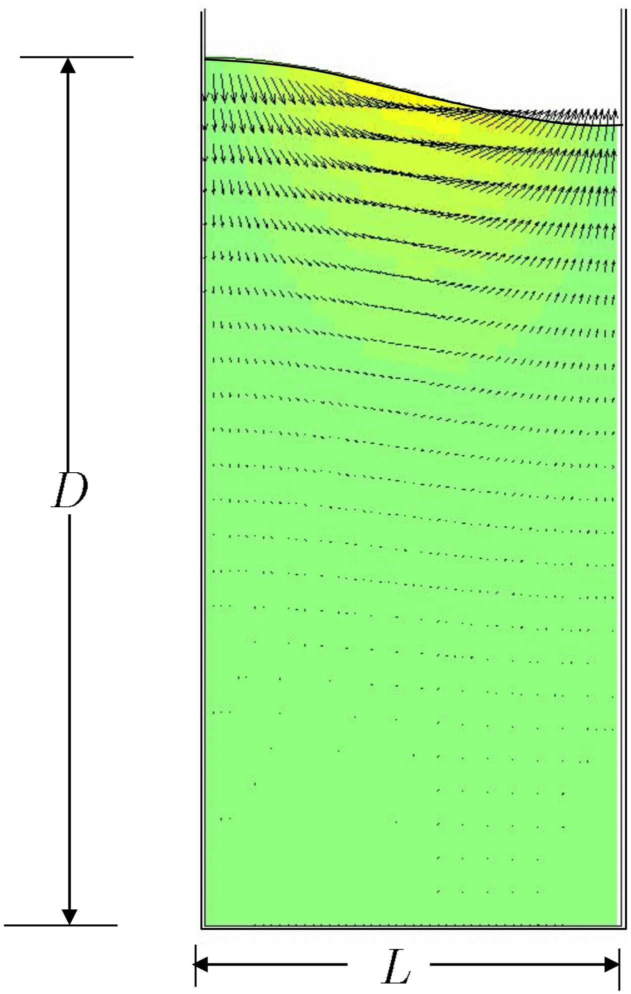
- Shallow-water limit $\varepsilon << 1$ (hydrostatic)

$$c^2 = gD$$



When is a flow nonhydrostatic?

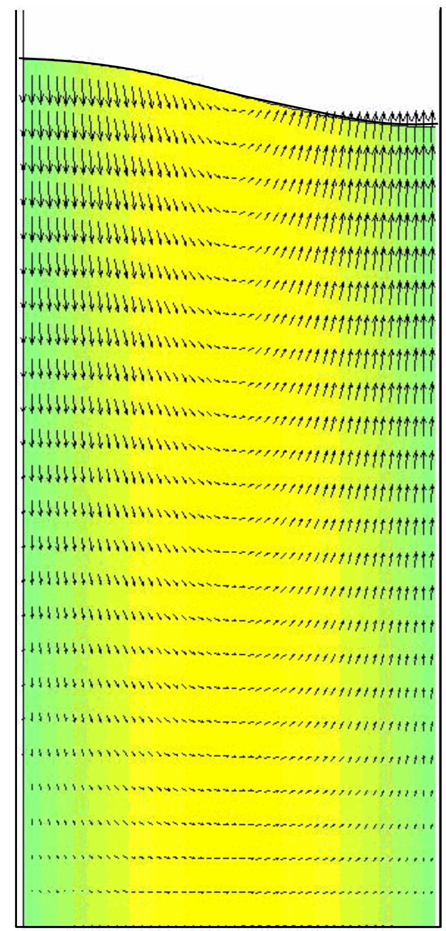
Nonhydrostatic Model



Aspect Ratio:

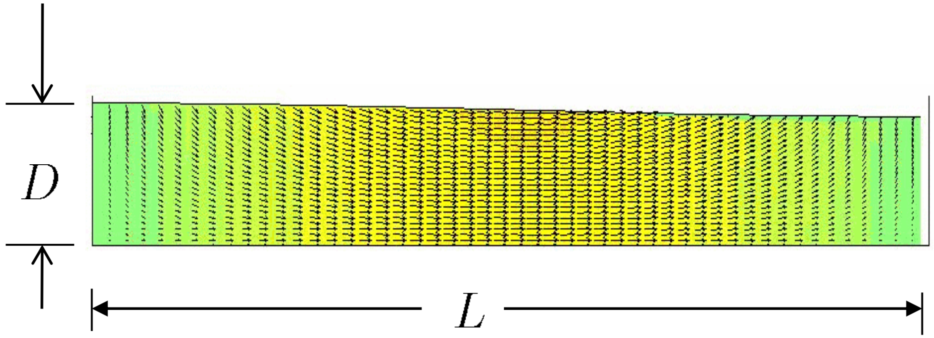
ε = D/L = 2

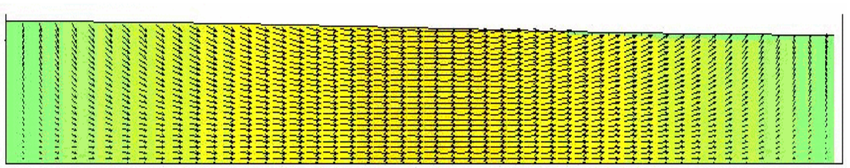
Hydrostatic Model



Nonhydrostatic Model

Hydrostatic Model





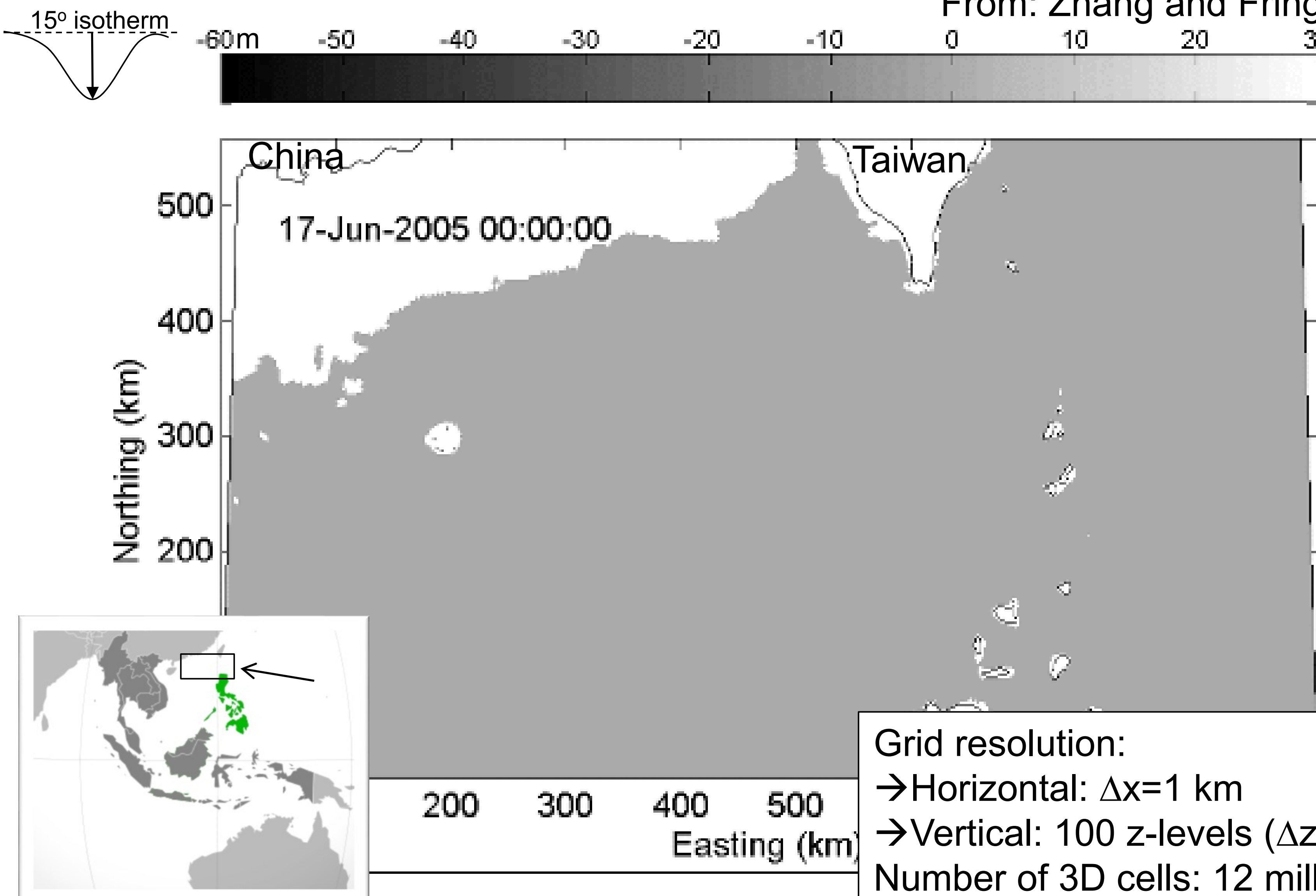
Aspect Ratio:

$\varepsilon = D/L = 1/8 = 0.125$

Nonhydrostatic result = Hydrostatic result + ε^2

Example 3D nonhydrostatic z-level simulation: Internal gravity waves in the South China Sea

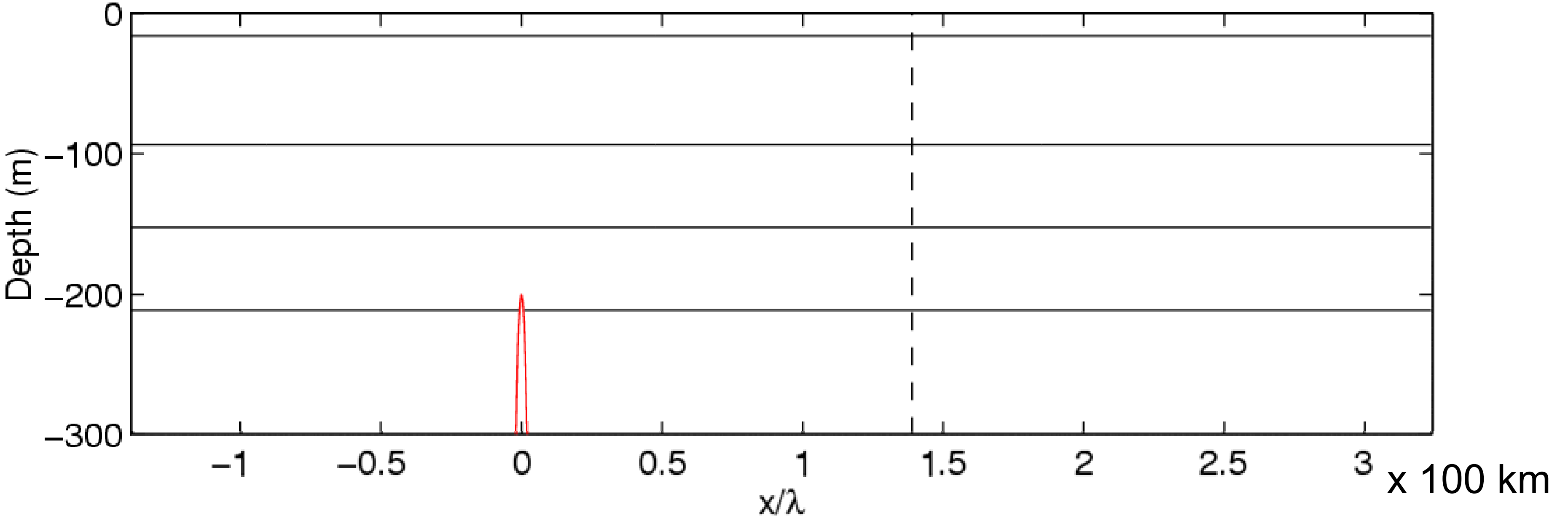
From: Zhang and Fringer (2011)



Grid resolution:
→Horizontal: Δx=1 km
→Vertical: 100 z-levels (Δz~10 m)
Number of 3D cells: 12 million

Generation of weakly nonlinear wavetrains

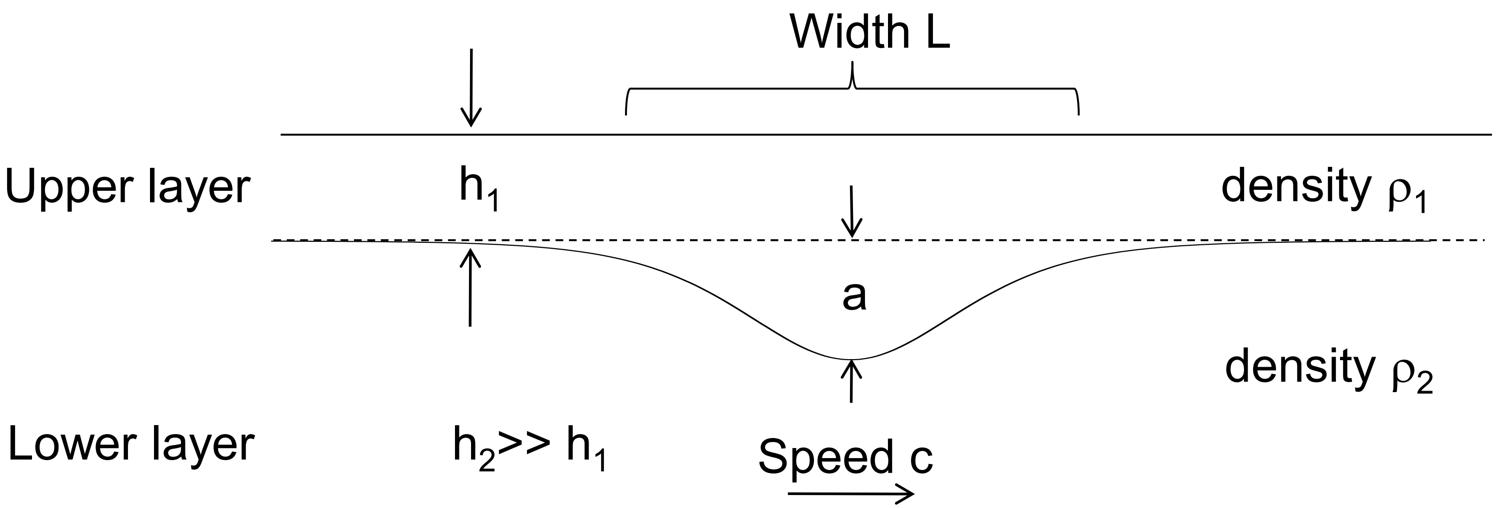
Isotherms: 16, 20, 24, 28 degrees C



Long internal tides O(100 km) → Short, solitary-like waves O(5 km)

How can we determine, apriori, how much horizontal grid resolution is needed to simulate this process?

Internal solitary waves



Nonlinear effect (steepening): $\delta = a/h_1$
Nonhydrostatic effect (frequency dispersion): $\varepsilon = h_1/L$

Solitary wave:

Balance between nonlinear steepening and nonhydrostatic dispersion.

$$\delta \sim \varepsilon^2$$

The KdV equation

When computing solitary waves, the behavior of a 3D, fully nonhydrostatic ocean model can be approximated very well with the KdV (Korteweg and de-Vries, 1895) equation:

Ocean Model:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p - \nabla q$$

Unsteadiness

Nonlinear momentum advection

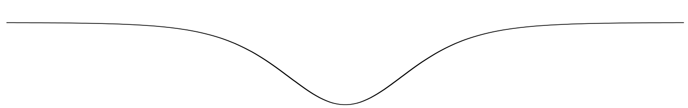
Hydrostatic pressure gradient.

Nonhydrostatic pressure gradient.

KdV:

$$\frac{\partial \xi}{\partial t} - \frac{3}{2}\delta\xi\frac{\partial \xi}{\partial x} = -\frac{\partial \xi}{\partial x} - \frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3}$$

$+O(\varepsilon^2\delta, \varepsilon^4)$



z = ξ

The KdV equation gives the well-known solution $\xi(x,t) = -a\,\mathrm{sech}^2\left(\frac{x-ct}{L_0}\right)$ $L_0 = \sqrt{\frac{4}{3}\frac{\varepsilon^2}{\delta a}}$

Numerical discretization of KdV

- Many ocean models discretize the equations with second-order accuracy in time and space.
(e.g. SUNTANS, Fringer et al. 2006; POM, Blumberg and Mellor, 1987; MICOM, Bleck et al., 1992; MOM, Pacanowski and Griffes, 1999).
- A second-order accurate discretization of the KdV equation using leapfrog (i.e. POM) is given by

$$\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x} + \frac{\epsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3} = 0$$

$$\frac{\xi_i^{n+1} - \xi_i^{n-1}}{2\Delta t} + \left(1 - \frac{3}{2}\delta\xi_i^n\right)\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} + \frac{\epsilon^2}{6}\frac{\frac{1}{2}\xi_{i+2}^n - \xi_{i+1}^n + \xi_{i-1}^n - \frac{1}{2}\xi_{i-2}^n}{(\Delta x)^3} = 0$$

- Use the Taylor series expansion to determine the modified equivalent form of the terms, e.g.

$$\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} = \left.\frac{\partial \xi}{\partial x}\right|_i^n + \frac{(\Delta x)^2}{6}\left.\frac{\partial^3 \xi}{\partial x^3}\right|_i^n + \frac{(\Delta x)^4}{120}\left.\frac{\partial^5 \xi}{\partial x^5}\right|_i^n + \frac{(\Delta x)^6}{5040}\left.\frac{\partial^7 \xi}{\partial x^7}\right|_i^n + O\left((\Delta x)^8\right)$$

Modified equivalent KdV equation

The discrete form of the KdV equation produces a solution to the modified equivalent PDE (Hirt 1968) which introduces new terms due to discretization errors:

KdV $$\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x} + \frac{\epsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3} = 0$$

Modified equivalent KdV:

$$\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x} + (1 + \Gamma)\frac{\epsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3} = O\left(\epsilon^2(\Delta x)^2, \delta(\Delta x)^2, (\Delta x)^4, (\Delta t)^4\right)$$

$$\Gamma = K\left(\frac{\Delta x}{h_1}\right)^2 = \frac{\text{Numerical dispersion}}{\text{Physical dispersion}} = K\lambda^2$$

$\lambda = \Delta x / h_1$ grid "lepticity" (Scotti and Mitran, 2008)

K=O(1) constant.

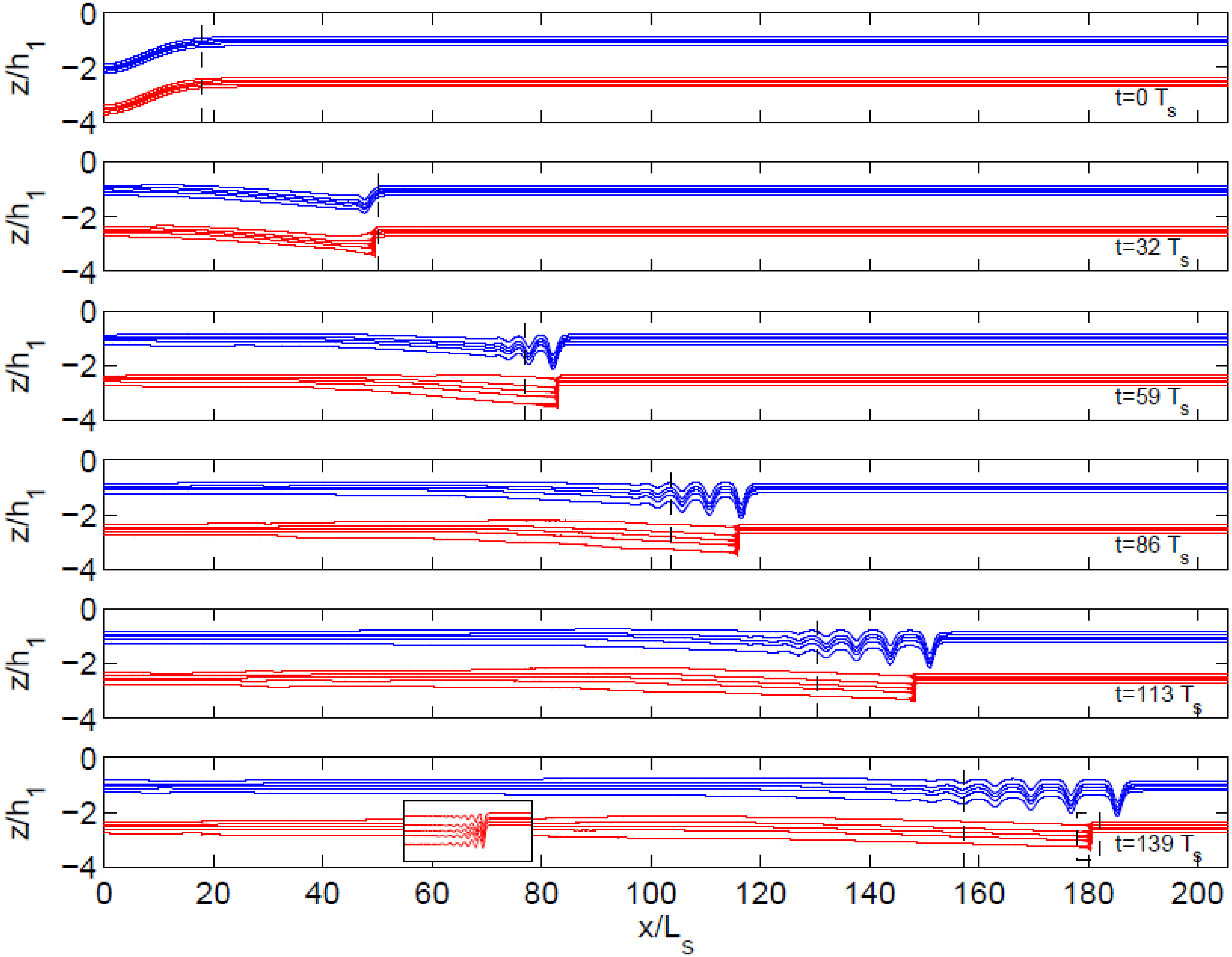
The numerical discretization of the first-order derivative produces numerical dispersion. Note that the errors in the nonlinear term are smaller by a factor δ.

For numerical dispersion to be smaller than physical dispersion, λ < 1.

Vitousek and Fringer (2011)

Hydrostatic vs. nonhydrostatic for λ=0.25

($\Delta x = h_1/4$)

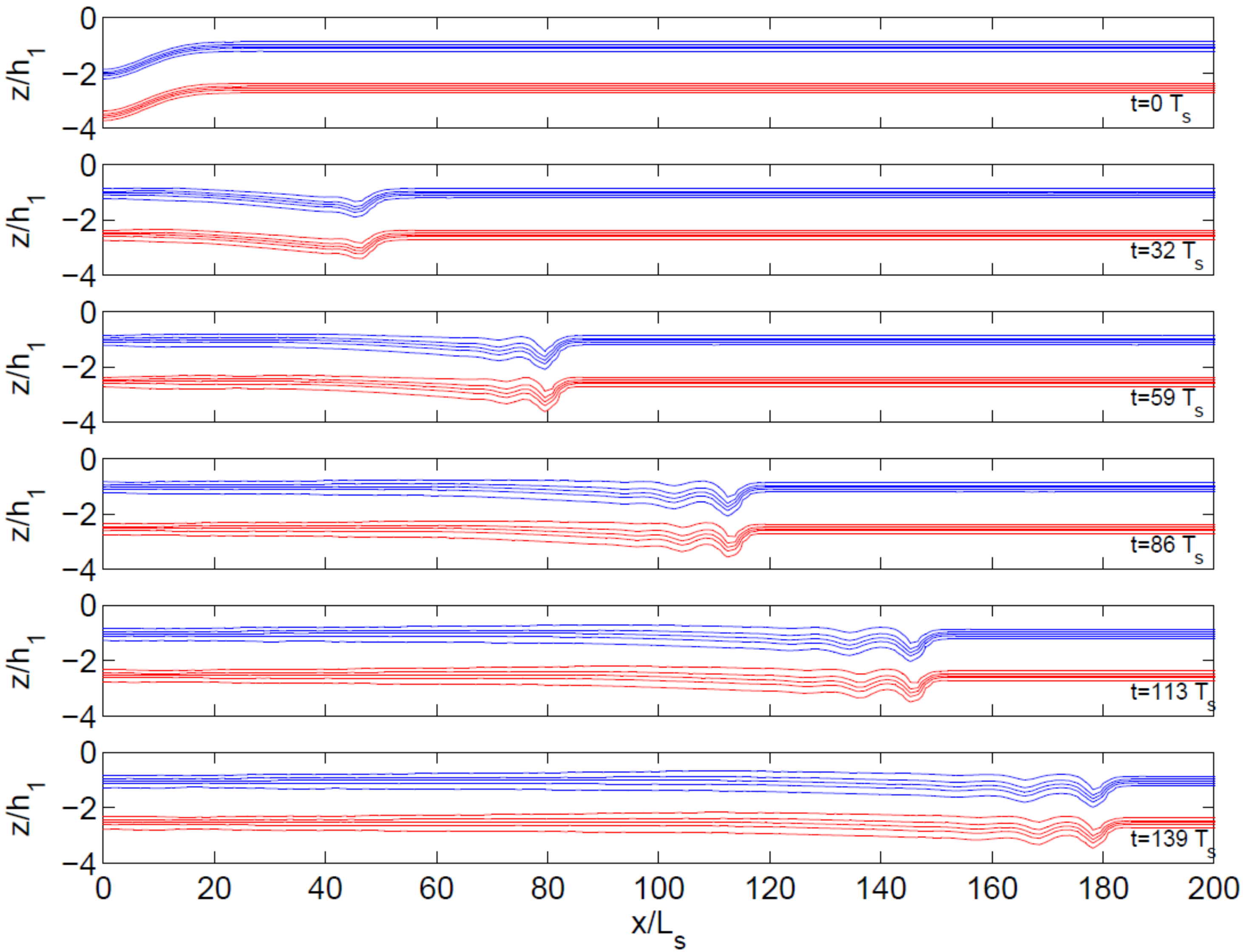


Numerical dispersion is 16 times smaller than physical dispersion.

Vitousek and Fringer (2011)

Hydrostatic vs. nonhydrostatic for $\lambda=8$

($\Delta x=8h_1$)



"Numerical solitary waves!"

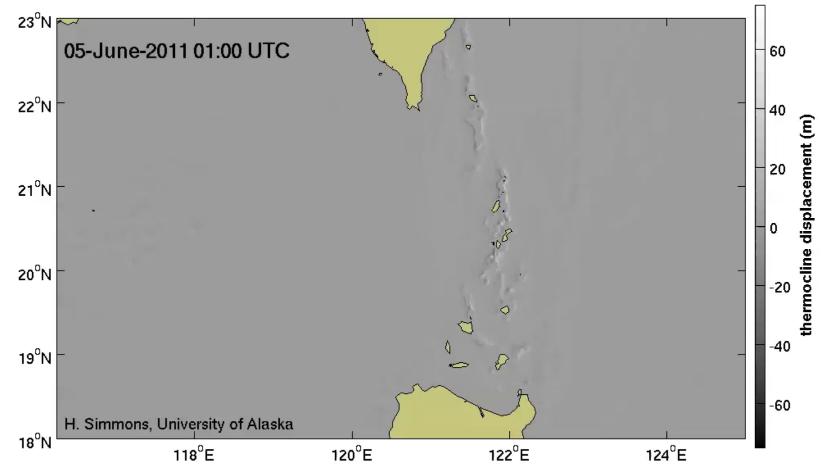
Numerical dispersion is 64 times larger than physical dispersion.

Vitousek and Fringer (2011)

Nonhydrostatic isopycnal model?

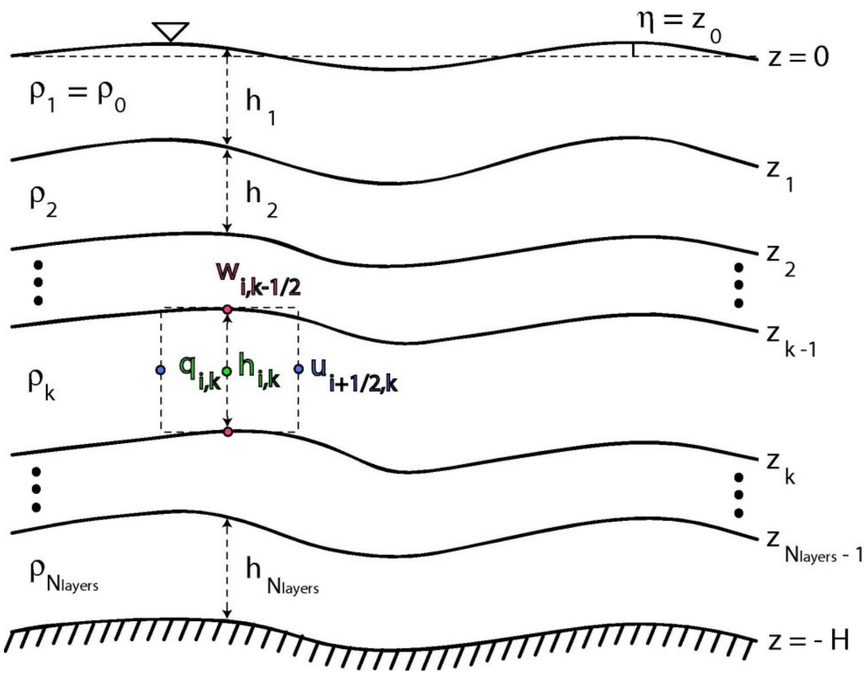
- Zhang et al. simulation:
  12 million cells, Δx=1 km = 5 h_1
- To ***begin*** to resolve nonhydrostatic effects, Δx=200 m = h_1 → 300 million cells! With Δx=100 m, 1.2 billion!
- The z-level SUNTANS model requires O(100) z-levels to minimize numerical diffusion of the pycnocline.
- Solution: Isopycnal model with O(2) layers
  = 50X reduction in computation time.

→ Nonhydrostatic isopycnal coordinate model.



2-layer hydrostatic result with isopycnal model of Simmons, U. Alaska Fairbanks.

Essential features of the nonhydrostatic isopycnal-coordinate model



- Staggered C-grid layout
- Split Montgomery potential into Barotropic (implicit) & Baroclinic (explicit) parts
- IMplicit-EXplicit (IMEX) multistep method. Durran (2012)
- MPDATA for upwinding of layer heights
- Implicit theta method for vertical diffusion
- Explicit horizontal diffusion
- Predictor/corrector method for nonhydrostatic pressure

→ Second-order accurate in time and space

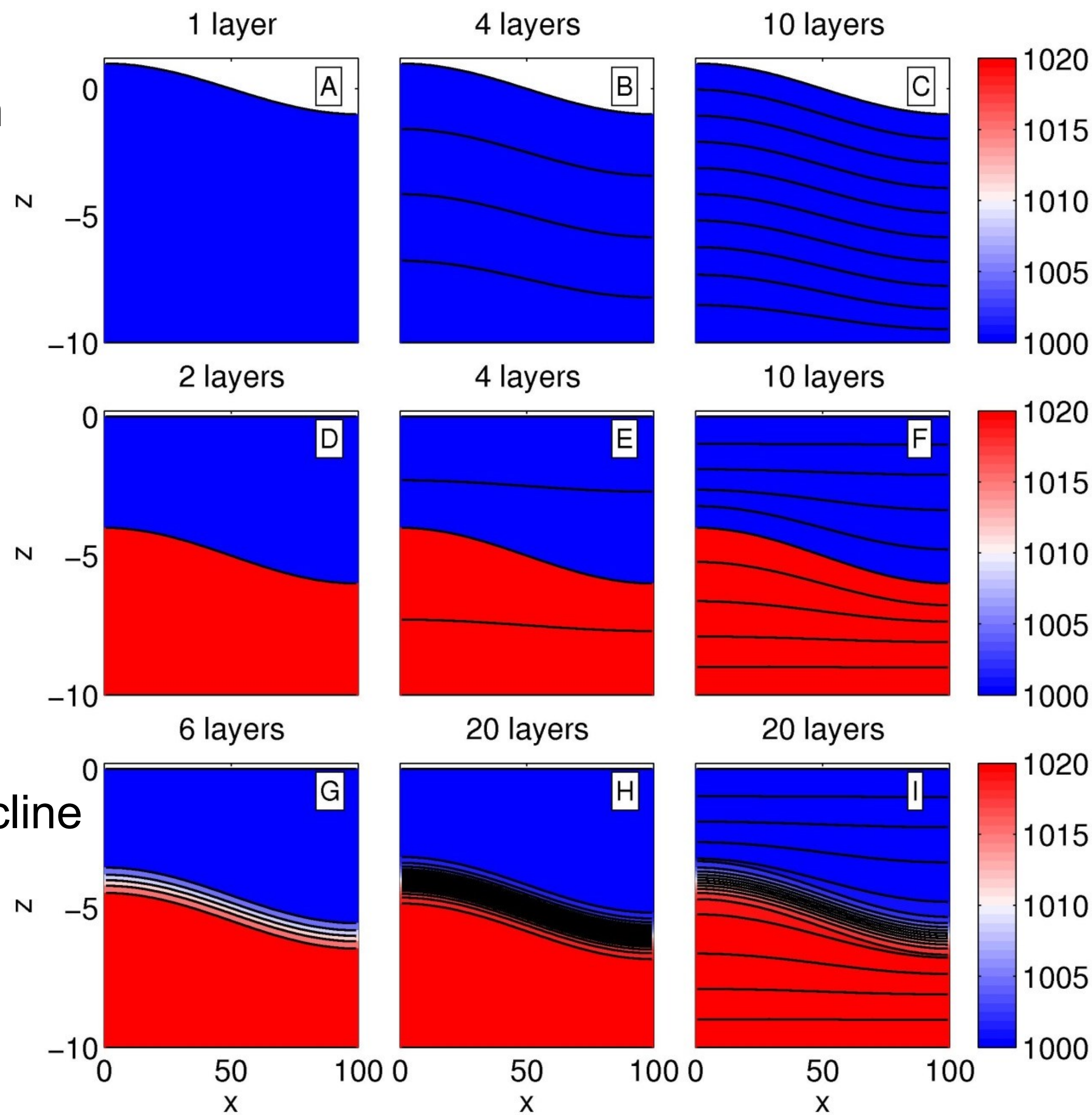
Vitousek and Fringer (2014)

Nonhydrostatic test cases

No stratification

Two-layer

Smooth pycnocline

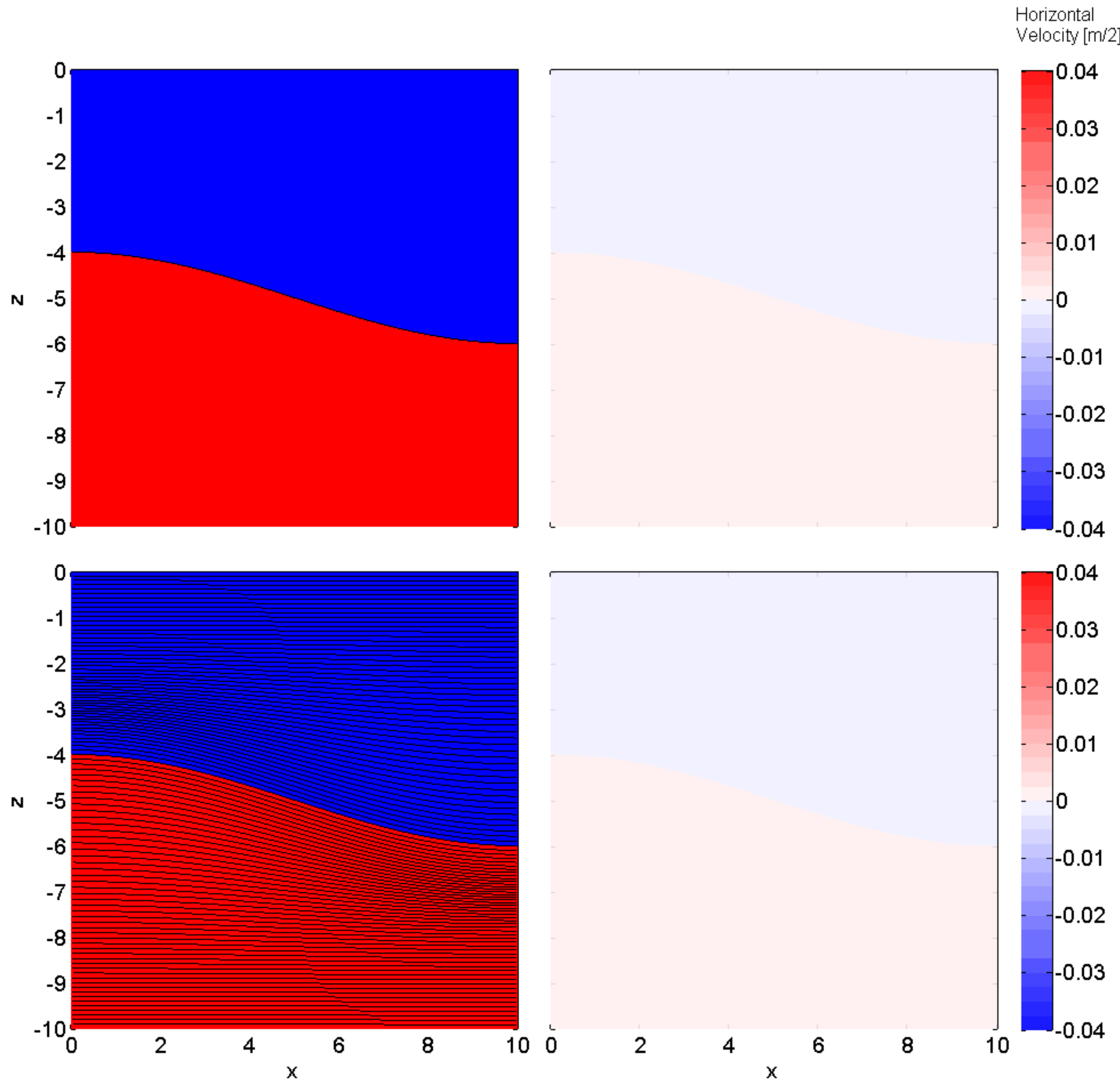


Note:
density need
not change
in each layer

Hydrostatic internal seiche

2 layers

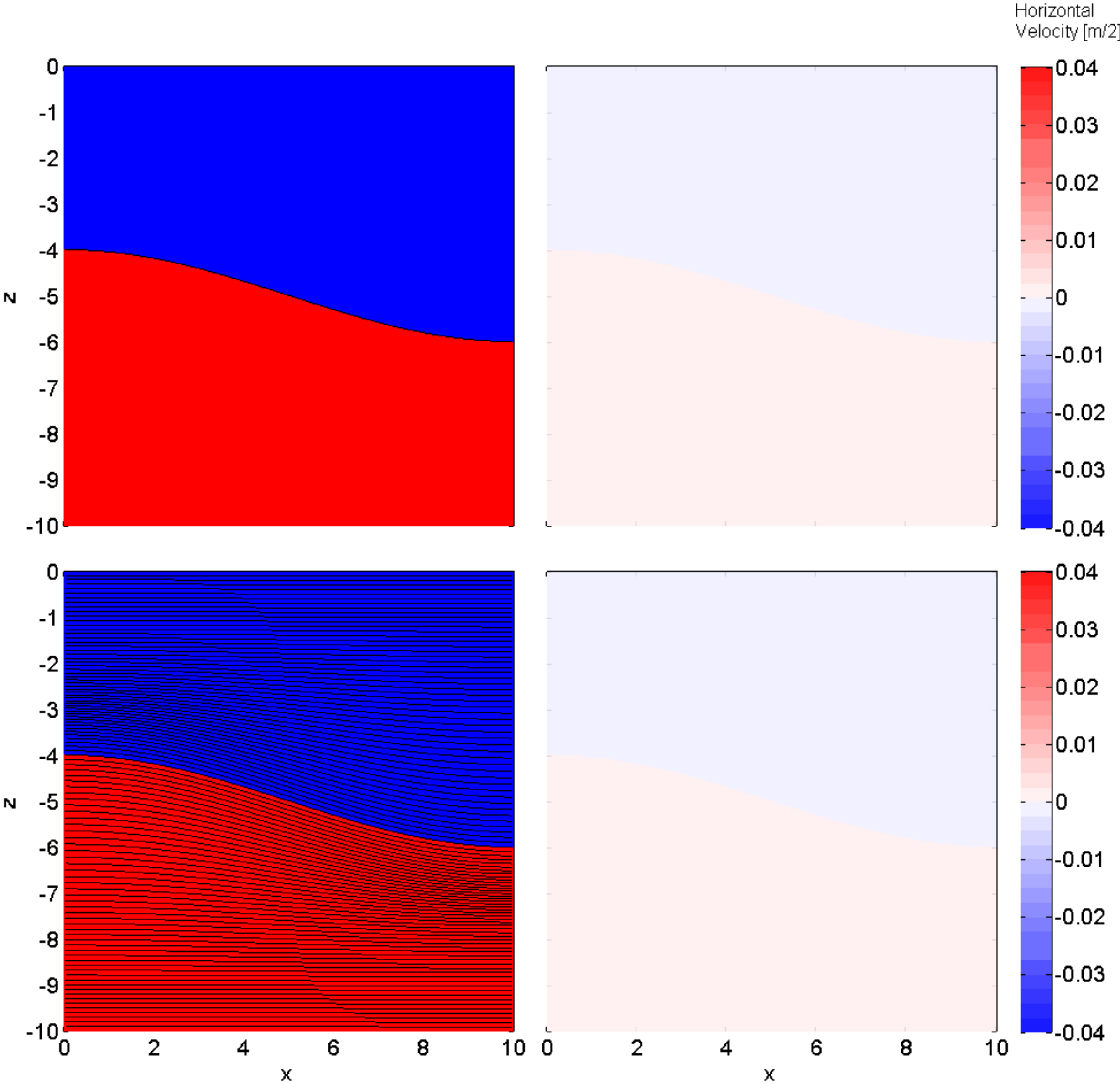
100 layers



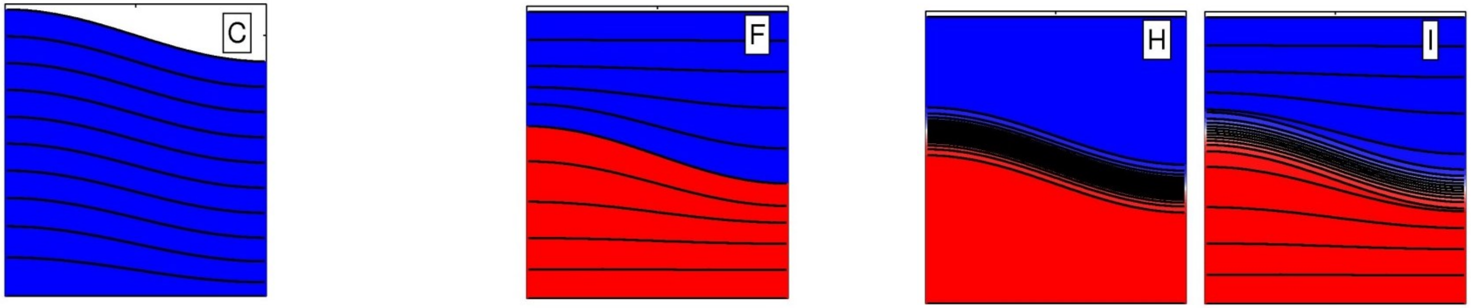
Nonhydrostatic internal seiche

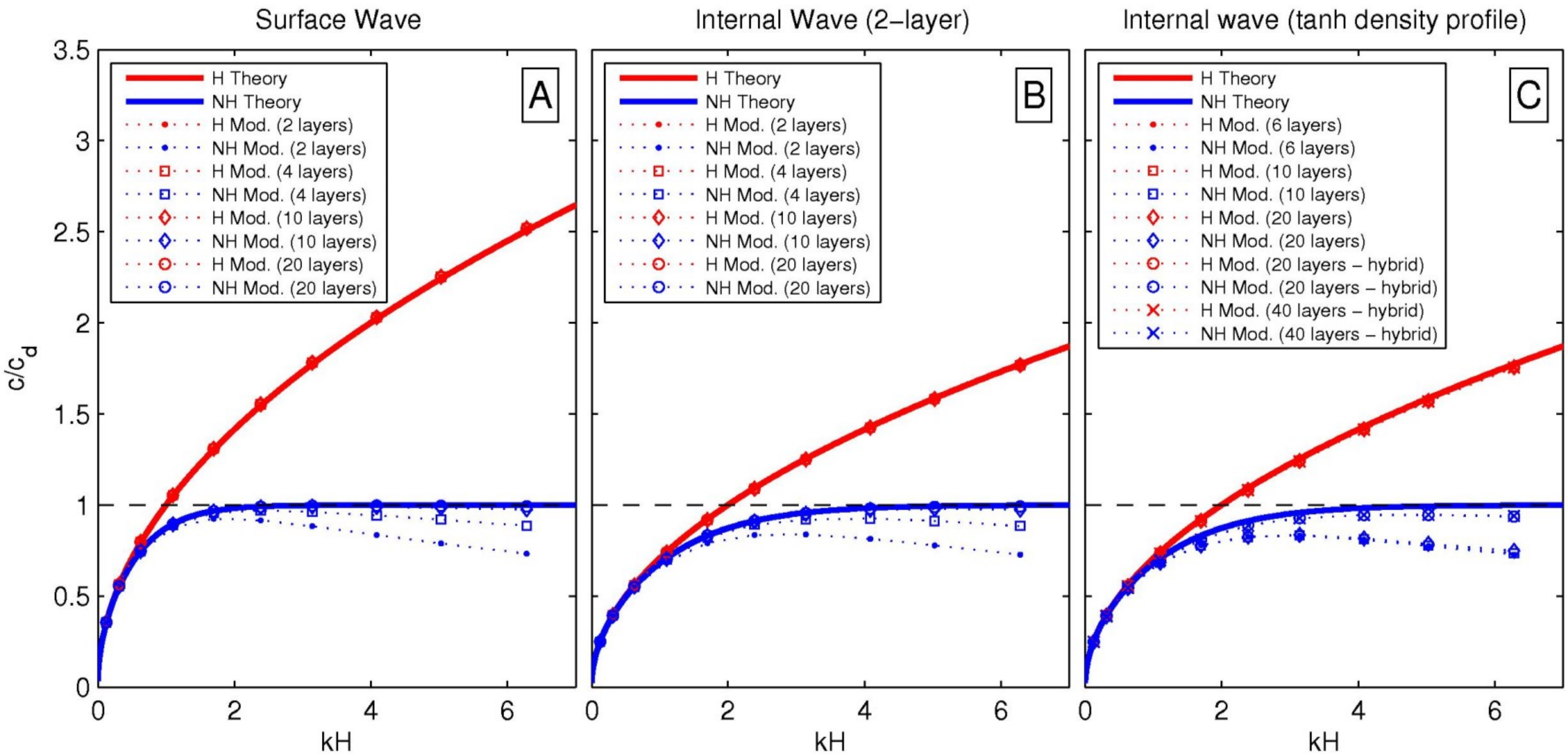
2 layers

100 layers



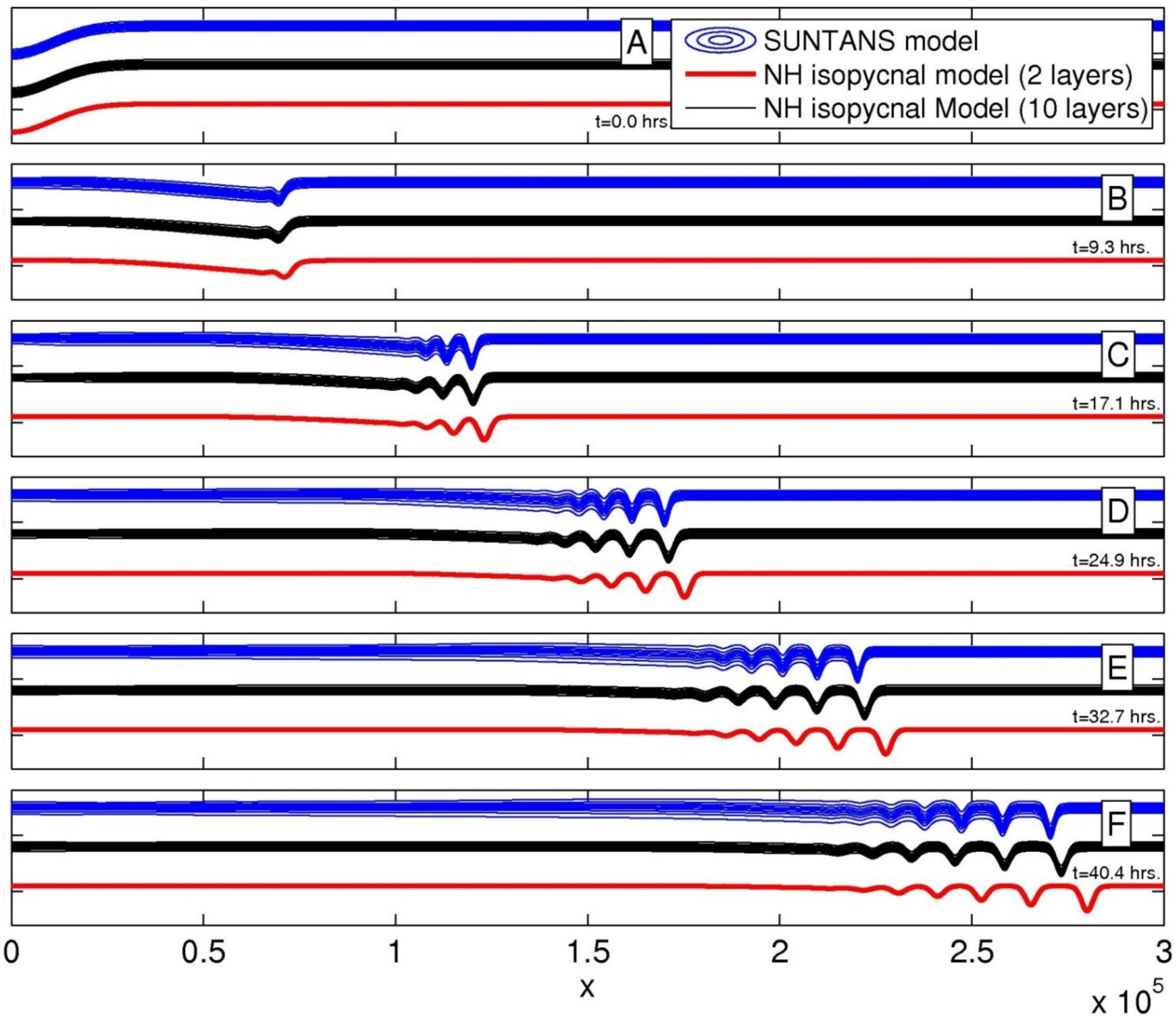
Dispersion relation
speed = function(wavelength)





Vitousek and Fringer (2014)

Internal solitary wave formation



z-level model leads to numerical diffusion, or thickening of the pycnocline.

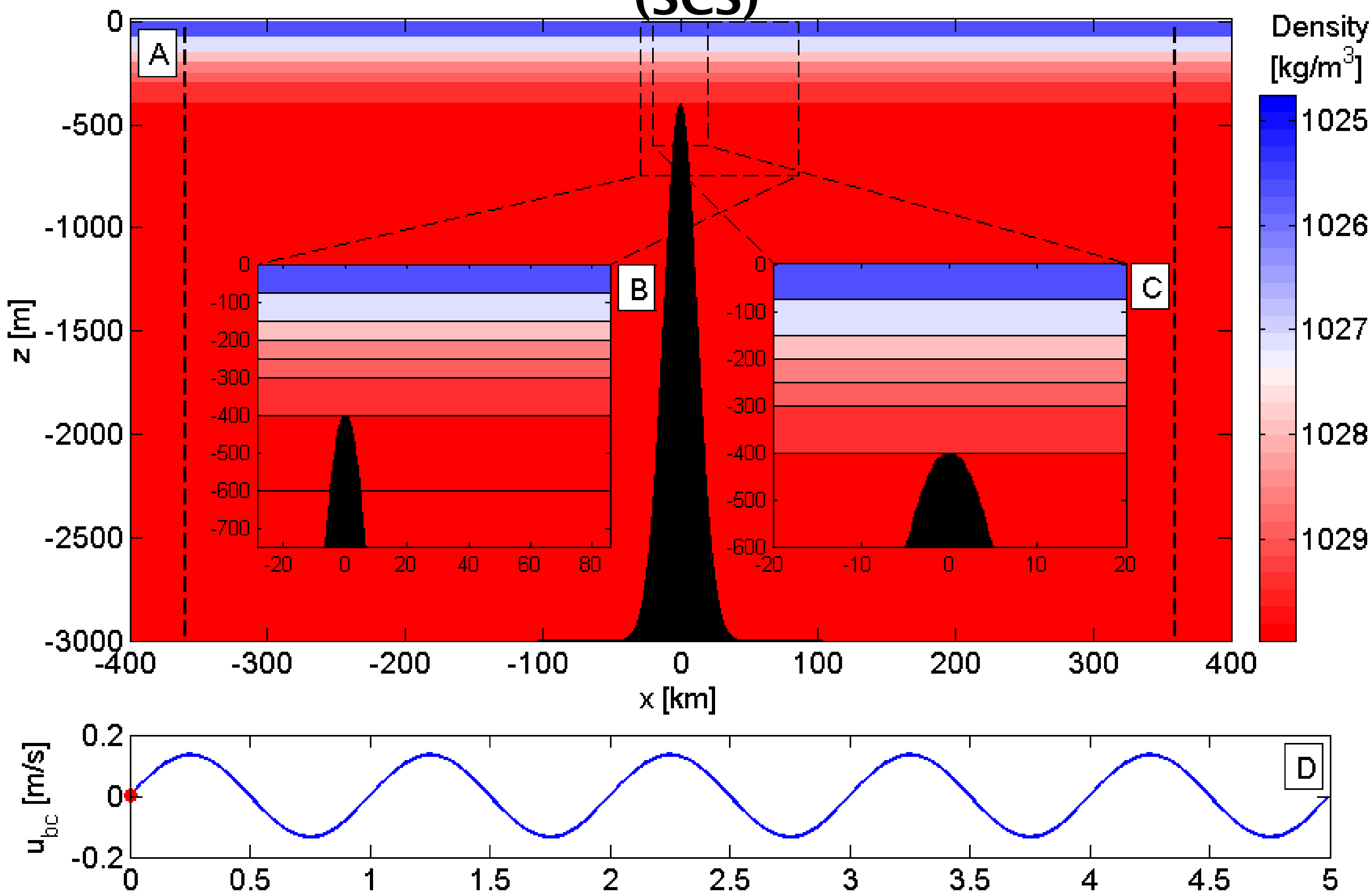
Isopycnal-coordinate model:

→ eliminates spurious numerical diffusion

→ captures solitary wave behavior at 1/50 cost…

Vitousek and Fringer (2014)

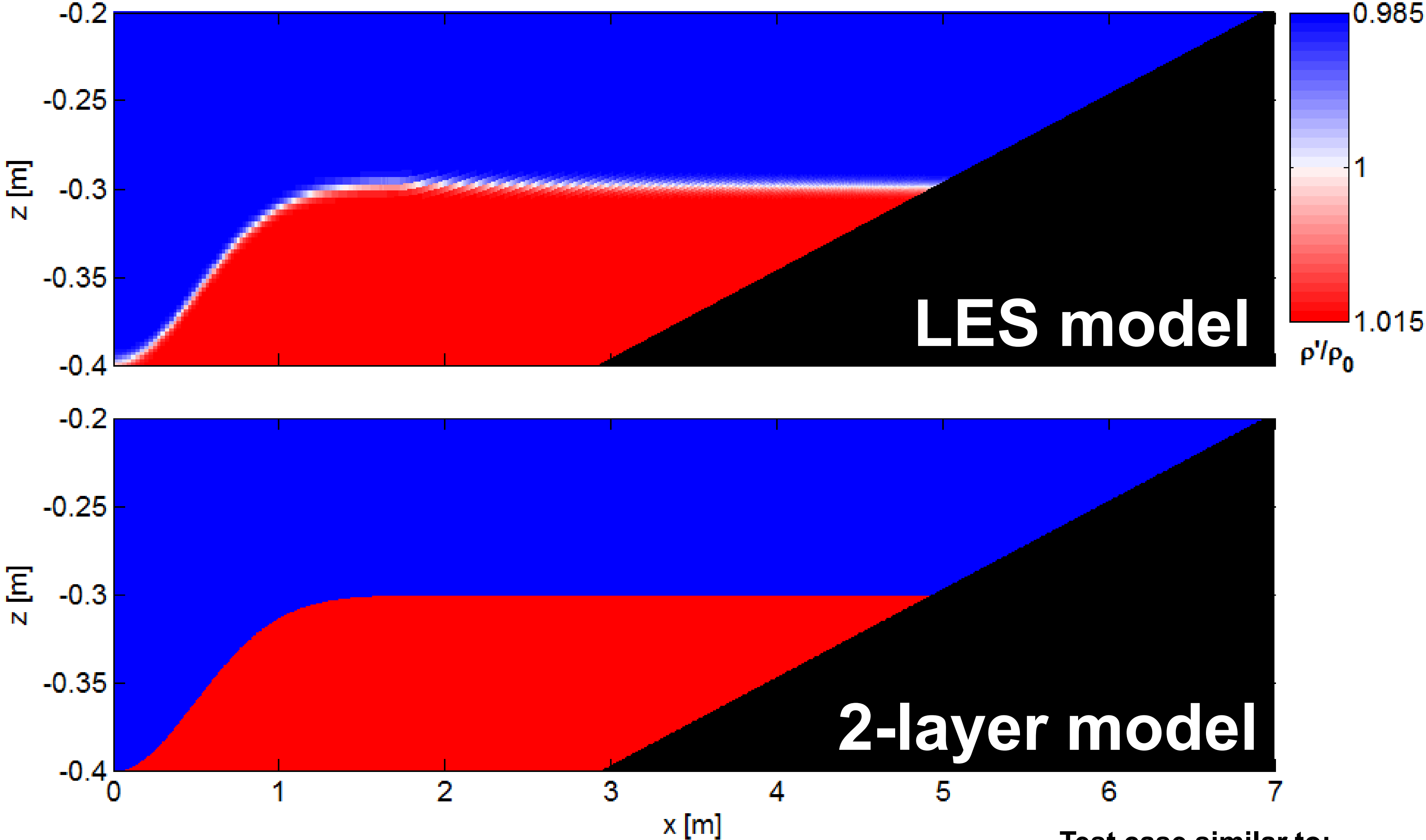
Internal Solitary Waves in the (idealized) South China Sea (SCS)



10-layer isopycnal model following Buijsman et al. (2010)

Vitousek and Fringer (2014)

Internal Wave Runup



2-layer isopycnal model vs. an LES model (Bobby Arthur, 2014)

Vitousek and Fringer (2014)

Test case similar to:
Michallet & Ivey 1999
Bourgault & Kelley 2004

Conclusions

- Simulation of nonhydrostatic effects in the SCS requires $\Delta x<h_1$ → O(billion) grid cells in 3D with z-level model.
- We have developed a nonhydrostatic isopycnal-coordinate model using stable higher-order time-stepping.
- More isopycnal layers are needed:
  - To resolve stratification
  - To resolve nonhydrostatic effects
- Most oceanic/lake processes are weakly nonhydrostatic and so <10 layers suffice for many applications. The result is a reduced computational cost by O(10).
- Ongoing work: Development of unstructured-grid model.